

ON THE ALGEBRAIC STRINGY EULER NUMBER

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ABSTRACT. We are interested in stringy invariants of singular projective algebraic varieties satisfying a strict monotonicity with respect to elementary birational modifications in the Mori program. We conjecture that the algebraic stringy Euler number is one of such invariants. In the present paper, we prove this conjecture for varieties having an action of a connected algebraic group G and admitting equivariant desingularizations with only finitely many G -orbits. In particular, we prove our conjecture for arbitrary projective spherical varieties.

1. INTRODUCTION

Let X be a smooth projective algebraic variety over \mathbb{C} . The dimension $b_{\text{top}}^i(X)$ of the i -th cohomology group $H^i(X, \mathbb{C})$ is called the i -th *topological Betti number* of X . The *topological Betti polynomial* of X is defined as

$$B_{\text{top}}(X, t) := \sum_{i=0}^{2 \dim X} (-1)^i b_{\text{top}}^i(X) t^i.$$

Its value

$$e_{\text{top}}(X) := B_{\text{top}}(X, 1) = \sum_{i=0}^{2 \dim X} (-1)^i b_{\text{top}}^i(X)$$

is the *topological Euler number* of X . Let $H_{\text{alg}}^{2i}(X, \mathbb{C})$ be the \mathbb{C} -subspace in $H^{2i}(X, \mathbb{C})$ generated by the classes $[Z]$ of algebraic cycles $Z \subseteq X$ of codimension i . We call the dimension $b_{\text{alg}}^{2i}(X)$ of the \mathbb{C} -space $H_{\text{alg}}^{2i}(X, \mathbb{C})$ the $2i$ -th *algebraic Betti number* of X . This naturally leads us to the *algebraic Betti polynomial*

$$B_{\text{alg}}(X, t) := \sum_{i=0}^{\dim X} b_{\text{alg}}^{2i}(X) t^{2i}.$$

Its value

$$e_{\text{alg}}(X) := B_{\text{alg}}(X, 1) = \sum_{i=0}^{\dim X} b_{\text{alg}}^{2i}(X)$$

we call the *algebraic Euler number* of X . Using the hard Lefschetz theorem, one can easily show that $e_{\text{alg}}(X) \geq \dim X + 1$. In particular, the algebraic Euler number $e_{\text{alg}}(X)$ of a smooth projective variety X is always a positive integer.

In order to consider stringy versions of the above invariants for singular varieties X we need some additional notations.

Let X be a normal projective algebraic variety over \mathbb{C} , and let K_X be its canonical class. A birational morphism $\rho: Y \rightarrow X$ is called a *log-desingularization* of X if Y is smooth and the exceptional locus of ρ consists of smooth irreducible divisors D_1, \dots, D_k with simple normal crossings. Assume that X is \mathbb{Q} -Gorenstein, i.e., some integral multiple of K_X is a Cartier divisor on X . We set $I := \{1, \dots, k\}$, $D_\emptyset := Y$,

and for any nonempty subset $J \subseteq I$ we denote by D_J the intersection of divisors $\bigcap_{j \in J} D_j$, which is either empty or a smooth projective subvariety in Y of codimension $|J|$. Then the canonical classes of X and Y are related by the formula

$$K_Y = \rho^* K_X + \sum_{i=1}^k a_i D_i,$$

where a_1, \dots, a_k are rational numbers which are called *discrepancies*. The singularities of X are said to be *log-terminal* if $a_i > -1$ for all $i \in I$.

For a projective algebraic variety with at worst \mathbb{Q} -Gorenstein log-terminal singularities one defines the *topological stringy Betti function*

$$B_{\text{top}}^{\text{str}}(X, t) := \sum_{\emptyset \subseteq J \subseteq I} B_{\text{top}}(D_J, t) \prod_{j \in J} \left(\frac{t^2 - 1}{t^{2(a_j+1)} - 1} - 1 \right)$$

and the *algebraic stringy Betti function*

$$B_{\text{alg}}^{\text{str}}(X, t) := \sum_{\emptyset \subseteq J \subseteq I} B_{\text{alg}}(D_J, t) \prod_{j \in J} \left(\frac{t^2 - 1}{t^{2(a_j+1)} - 1} - 1 \right).$$

In particular, one obtains the *topological stringy Euler number*

$$e_{\text{top}}^{\text{str}}(X) := \lim_{t \rightarrow 1} B_{\text{top}}^{\text{str}}(X, t) = \sum_{\emptyset \subseteq J \subseteq I} e_{\text{top}}(D_J) \prod_{j \in J} \left(\frac{-a_j}{a_j + 1} \right)$$

and the *algebraic stringy Euler number*

$$e_{\text{alg}}^{\text{str}}(X) := \lim_{t \rightarrow 1} B_{\text{alg}}^{\text{str}}(X, t) = \sum_{\emptyset \subseteq J \subseteq I} e_{\text{alg}}(D_J) \prod_{j \in J} \left(\frac{-a_j}{a_j + 1} \right).$$

It is well-known that the definition of $B_{\text{top}}^{\text{str}}(X, t)$ and $e_{\text{top}}^{\text{str}}(X)$ does not depend on the choice of a desingularization ρ (see [Bat98]). The proof of this statement uses a non-archimedean integration and the fact that the Betti polynomial $B(X, t)$ can be extended to an additive function on the category $\mathcal{V}_{\mathbb{C}}$ of algebraic varieties over \mathbb{C} having the multiplicativity property:

$$B_{\text{top}}(X_1 \times X_2, t) = B_{\text{top}}(X_1, t) \cdot B_{\text{top}}(X_2, t) \quad \forall X_1, X_2 \in \mathcal{V}_{\mathbb{C}}.$$

We note that the algebraic Betti polynomial $B_{\text{alg}}(X, t)$ can also be extended to an additive function on the category of arbitrary algebraic varieties, but the above multiplicativity property fails for $B_{\text{alg}}(X, t)$ in general. Nevertheless, it was noticed in [Teh09] that the weaker multiplicativity property

$$B_{\text{alg}}(X \times \mathbb{P}^1, t) = B_{\text{alg}}(X, t) \cdot B_{\text{alg}}(\mathbb{P}^1, t) \quad \forall X \in \mathcal{V}_{\mathbb{C}}$$

is sufficient in order to define stringy invariants. In particular, the algebraic stringy Betti function $B_{\text{alg}}^{\text{str}}(X, t)$ and the stringy algebraic Euler number $e_{\text{alg}}^{\text{str}}(X)$ do not depend on the choice of a desingularization ρ .

The algebraic stringy Euler number $e_{\text{alg}}^{\text{str}}(X)$ of a singular variety X is usually a rational number. Our first conjecture claims the positivity of the algebraic stringy Euler number:

Conjecture 1.1. *Let X be a projective algebraic variety with at worst \mathbb{Q} -Gorenstein log-terminal singularities. Then $e_{\text{alg}}^{\text{str}}(X) > 0$.*

Remark 1.2. It is natural to expect an even stronger inequality:

$$e_{\text{alg}}^{\text{str}}(X) \geq \dim X + 1.$$

Now we consider some relations between the stringy Euler numbers and the birational geometry.

Recall that two smooth birationally isomorphic projective varieties X_1 and X_2 are called K -equivalent if there exists a birational isomorphism $f: X_1 \dashrightarrow X_2$ and a smooth projective variety Y together with two birational morphisms $\rho_1: Y \rightarrow X_1$ and $\rho_2: Y \rightarrow X_2$ in a commutative diagram

$$\begin{array}{ccc} & Y & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ X_1 & \dashrightarrow^f & X_2 \end{array}$$

such that the pullbacks of the canonical classes of X_1 and X_2 are the same: $\rho_1^*(K_{X_1}) = \rho_2^*(K_{X_2})$. Using methods of a non-archimedean integration, one can show that K -equivalent algebraic varieties must have the same Betti polynomials: $B_{\text{top}}(X_1, t) = B_{\text{top}}(X_2, t)$. In particular, two birationally isomorphic Calabi-Yau manifolds must have the same Betti numbers (see [Bat99a]).

The notion of K -equivalence immediately extends to algebraic varieties with at worst \mathbb{Q} -Gorenstein singularities. Using the same method based on a non-archimedean integration, one can prove that two K -equivalent projective varieties X_1 and X_2 having at worst log-terminal singularities must have the same stringy Betti functions: $B_{\text{top}}^{\text{str}}(X_1, t) = B_{\text{top}}^{\text{str}}(X_2, t)$. In particular, one has the equality for the stringy Euler numbers $e_{\text{top}}^{\text{str}}(X_1) = e_{\text{top}}^{\text{str}}(X_2)$ if X_1 and X_2 are K -equivalent.

Definition 1.3. A proper birational morphism $f: X \rightarrow X'$ of two \mathbb{Q} -Gorenstein varieties X and X' is called a *divisorial Mori contraction* if f contracts a divisor $D \subseteq X$ and the anticanonical divisor $-K_X$ is f -ample.

Definition 1.4. A birational morphism $g: X \dashrightarrow X^+$ of two \mathbb{Q} -Gorenstein varieties X and X^+ together with the birational morphisms $f: X \rightarrow Z$ and $f^+: X^+ \rightarrow Z$ in a commutative diagram

$$\begin{array}{ccc} X & \dashrightarrow^g & X^+ \\ & \searrow f & \swarrow f^+ \\ & Z & \end{array}$$

is called a *Mori flip* if g is an isomorphism in codimension one, the anticanonical class $-K_X$ is f -ample, and the canonical class K_{X^+} is f^+ -ample.

We conjecture the following strict monotonicity of the algebraic stringy Euler number with respect to the above elementary birational transformations in the Mori program:

Conjecture 1.5. *Let $f: X \rightarrow X'$ be a divisorial Mori contraction. Then*

$$e_{\text{alg}}^{\text{str}}(X) > e_{\text{alg}}^{\text{str}}(X').$$

Conjecture 1.6. *Let $g: X \dashrightarrow X^+$ be a Mori flip. Then*

$$e_{\text{alg}}^{\text{str}}(X) > e_{\text{alg}}^{\text{str}}(X^+).$$

Remark 1.7. From the viewpoint of Conjectures 1.1, 1.5, and 1.6, a projective algebraic variety X with at worst terminal singularities is a *minimal model* in a given birational class if its algebraic stringy Euler number $e_{\text{alg}}^{\text{str}}(X)$ has the minimal possible value among all projective algebraic varieties with at worst terminal singularities in the same birational class.

Since $e_{\text{top}}(X) = e_{\text{top}}^{\text{str}}(X)$ for smooth algebraic varieties, Conjecture 1.1 is obviously false for the topological stringy Euler number $e_{\text{top}}^{\text{str}}(X)$, because $e_{\text{top}}(C) < 0$ already for smooth projective algebraic curves C of genus $g \geq 2$. If X is a blow-up of a smooth curve C in a smooth minimal threefold X' , then $e_{\text{top}}(X) = e_{\text{top}}(X') + e_{\text{top}}(C)$ and one gets the opposite inequality $e_{\text{top}}^{\text{str}}(X) < e_{\text{top}}^{\text{str}}(X')$ if $g \geq 2$. Therefore Conjecture 1.5 is also false for topological stringy Euler numbers. This does not happen for the algebraic Euler number $e_{\text{alg}}(X)$ of smooth projective varieties X , because the cohomology groups in odd degrees do not contribute to it.

In general, the topological and algebraic stringy Euler numbers of a singular variety are rational numbers. However, if $X = X_{\Sigma}$ is a d -dimensional \mathbb{Q} -Gorenstein projective toric variety associated to a rational polyhedral fan Σ in a d -dimensional \mathbb{Q} -space $N_{\mathbb{Q}}$, then the stringy Euler number $e_{\text{top}}^{\text{str}}(X_{\Sigma})$ is always a positive integer. This statement follows from the explicit computation of the stringy Betti function of the toric variety X_{Σ} by the formula:

$$B_{\text{top}}^{\text{str}}(X_{\Sigma}, t) = (t^2 - 1)^d \sum_{n \in N} t^{-2\kappa(n)},$$

where $N \subseteq N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ is the lattice of 1-parameter subgroups in the d -dimensional algebraic torus acting on X_{Σ} and $\kappa: N_{\mathbb{Q}} \rightarrow \mathbb{Q}$ is a special piecewise linear function having the following properties: 1) κ is linear on every cone $\sigma \in \Sigma$; 2) κ has value 1 on primitive lattice generators of 1-dimensional cones in Σ (see [Bat98]). Moreover, it was observed in [Bat98, Proposition 4.10] that the stringy Euler number $e_{\text{top}}^{\text{str}}(X_{\Sigma})$ of a d -dimensional toric variety X_{Σ} coincides with the normalized volume $d! \text{vol}(\text{shed}(\Sigma))$ of the bounded polyhedral set

$$\text{shed}(\Sigma) := \{x \in N_{\mathbb{Q}} \mid \kappa(x) \leq 1\},$$

which is called the *shed of the fan* Σ (see [Rei83, (4.2)]). Torus-equivariant birational morphisms of toric varieties $X_{\Sigma} \dashrightarrow X_{\Sigma'}$ can be described by polyhedral subdivisions of the corresponding fans. It was observed by Reid in [Rei83] that one has strict inclusions $\text{shed}(\Sigma') \subsetneq \text{shed}(\Sigma)$ for divisorial toric Mori contractions $f: X_{\Sigma} \rightarrow X_{\Sigma'}$ and $\text{shed}(\Sigma^+) \subsetneq \text{shed}(\Sigma)$ for toric Mori flips $g: X_{\Sigma} \dashrightarrow X_{\Sigma^+}$. Therefore, the statements in Conjectures 1.5, 1.6 hold true for equivariant birational morphism in the toric Mori program. It is not difficult to show that for projective toric varieties X one always has even a stronger inequality $e_{\text{top}}^{\text{str}}(X) \geq \dim X + 1$.

Let G be a connected reductive algebraic group. An irreducible normal G -variety X is called *spherical* if a Borel subgroup $B \subseteq G$ has an open orbit in X . Spherical varieties are generalizations of toric varieties. Following the equivariant Mori program for toric varieties in [Rei83], the equivariant Mori program for spherical varieties was considered by Brion and Knop in [Bri93, BK94] (see also the recent papers of Perrin

and Pasquier [Per14, Pas15a]). It is known that a \mathbb{Q} -Gorenstein spherical variety is always log-terminal (see [AB04] and [Pas15c, Proposition 5.6]). As in the case of toric varieties, one has the equality $B_{\text{alg}}^{\text{str}}(X, t) = B_{\text{top}}^{\text{str}}(X, t)$ for any spherical variety X . However, the stringy Euler number of a projective spherical variety X is not always an integer even in the case when X is Gorenstein (see [BM13, Example 4.6]).

Our main purpose of this paper is to attract attention to the above conjectures and to prove them for arbitrary projective spherical varieties:

Theorem 1.8. *Let G be a connected reductive algebraic group. Conjectures 1.1, 1.5 and 1.6 are true in the G -equivariant Mori program for projective spherical G -varieties.*

We remark that our proof of the above theorem works even for more general G -varieties having a regular action of a connected algebraic group G . We need only the following well-known properties of spherical varieties: 1) every spherical G -variety X contains only finitely many G -orbits; 2) every singular spherical G -variety X always admits a G -equivariant desingularization Y which is again a spherical variety; 3) every birational G -equivariant map $g: X_1 \dashrightarrow X_2$ between two spherical G -varieties can be extended to a G -equivariant commutative diagram

$$\begin{array}{ccc} & Y & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ X_1 & \overset{f}{\dashrightarrow} & X_2 \end{array}$$

for some smooth spherical G -variety Y and G -equivariant log-desingularizations $\rho_1: Y \rightarrow X_1$ and $\rho_2: Y \rightarrow X_2$.

In Section 2, we explain the construction of generalized stringy invariants of algebraic varieties that include the algebraic stringy Euler number and some other invariants that could attain minimum on the minimal models in a given birational class.

In Section 3, we prove Theorem 1.8 as a consequence of some more general results on the G -equivariant birational Mori theory. We also discuss some generalizations of our results that include the G -equivariant Mori program for pairs (X, Δ) .

2. GENERALIZED STRINGY INVARIANTS OF ALGEBRAIC VARIETIES

We denote by $\mathcal{V}_{\mathbb{C}}$ the category of complex algebraic varieties. Let $K_0(\mathcal{V}_{\mathbb{C}})$ be the *Grothendieck group of complex algebraic varieties*, which is the quotient of the free abelian group generated by isomorphism classes $[X]$ of complex algebraic varieties modulo the subgroup generated by $[X] - [X'] - [X \setminus X']$ for closed subvarieties $X' \subseteq X$. One defines the structure of a commutative ring on $K_0(\mathcal{V}_{\mathbb{C}})$ by the equation $[X] \cdot [X'] := [X \times X']$. We can consider the ring $K_0(\mathcal{V}_{\mathbb{C}})$ as a module over the polynomial ring $\mathbb{Z}[\lambda]$ using the ring homomorphism $\mathbb{Z}[\lambda] \rightarrow K_0(\mathcal{V}_{\mathbb{C}})$ that maps λ to the class $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathcal{V}_{\mathbb{C}})$.

Let $S := S(\lambda) \subseteq \mathbb{Z}[\lambda]$ be the multiplicative subset generated by λ and the polynomials $\sum_{i=0}^k \lambda^i$ for all $k \geq 0$. We define a filtration F_{∞}^* on the localized ring $S^{-1}\mathbb{Z}[\lambda]$ using the subgroups $F_{\infty}^m S^{-1}\mathbb{Z}[\lambda]$ ($m \in \mathbb{Z}$) generated by all rational functions $p(\lambda)/q(\lambda)$ such that $q(\lambda) \in S$ and $\deg q - \deg p \geq m$. This filtration is obtained by

the restriction to $S^{-1}\mathbb{Z}[\lambda]$ of the filtration on the ring of rational functions on \mathbb{P}^1 over $\text{Spec } \mathbb{Z}$ defined by the discrete valuation corresponding to the point $\infty \in \mathbb{P}^1$.

We define the filtration \mathcal{F}^* on the $S^{-1}\mathbb{Z}[\lambda]$ -algebra

$$\widetilde{\mathcal{M}}_{\mathbb{C}} := S^{-1}K_0(\mathcal{V}_{\mathbb{C}})$$

where $\mathcal{F}^m \widetilde{\mathcal{M}}_{\mathbb{C}}$ is the subgroup of $\widetilde{\mathcal{M}}_{\mathbb{C}}$ generated by

$$\left\{ \frac{[X]}{q(\mathbb{L})} : \deg q(\lambda) - \dim X \geq m, q(\lambda) \in S \right\}.$$

This filtration is compatible with the filtration F_{∞}^* on $S^{-1}\mathbb{Z}[\lambda]$:

$$F_{\infty}^m \cdot \mathcal{F}^l \subseteq \mathcal{F}^{m+l} \quad \forall m, l \in \mathbb{Z}.$$

In papers devoted to motivic integration (see, for instance, [DL99, (3.2)] or [Cra04, Definition 2.11]) one often considers the ring $\widehat{\mathcal{M}}_{\mathbb{C}}$ obtained as the completion of the ring $\widetilde{\mathcal{M}}_{\mathbb{C}}$ with respect to the filtration \mathcal{F}^* . However, the kernel of the completion map $\mathcal{M}_{\mathbb{C}} \rightarrow \widehat{\mathcal{M}}_{\mathbb{C}}$ is rather difficult to understand, because $\mathcal{M}_{\mathbb{C}}$ is not a noetherian ring. The above filtration is necessary for the definition of the topology and the convergence in this topology, but we do not really need the completed ring $\widehat{\mathcal{M}}_{\mathbb{C}}$, because all considered non-archimedean motivic integrals will take their values in the localized ring $\widetilde{\mathcal{M}}_{\mathbb{C}}$ or in an extension $\widetilde{\mathcal{M}}_{\mathbb{C}}^{(r)}$, which we explain next.

Let r be a positive integer. Using the ring monomorphism

$$\iota_r: S^{-1}\mathbb{Z}[\lambda] \rightarrow S^{-1}\mathbb{Z}[\lambda], \quad \lambda \mapsto \lambda^r,$$

we can identify the image of ι_r with $S^{-1}\mathbb{Z}[\lambda]$ and obtain the ring extension

$$S^{-1}\mathbb{Z}[\lambda] \subseteq S(\lambda^{1/r})^{-1}\mathbb{Z}[\lambda^{1/r}].$$

This allows to consider any rational function

$$\frac{\lambda - 1}{\lambda^{k/r} - 1}, \quad k \in \mathbb{N},$$

as an element of $S(\lambda^{1/r})^{-1}\mathbb{Z}[\lambda^{1/r}]$. For any $\mathbb{Z}[\lambda]$ -module M , we define

$$S_{(r)}^{-1}M := S(\lambda^{1/r})^{-1}\mathbb{Z}[\lambda^{1/r}] \otimes_{S^{-1}\mathbb{Z}[\lambda]} S^{-1}M,$$

and we define

$$\widetilde{\mathcal{M}}_{\mathbb{C}}^{(r)} := S_{(r)}^{-1}K_0(\mathcal{V}_{\mathbb{C}}).$$

The following Definition 2.1 and Theorem 2.2 are slightly modified versions of some of the content from [Teh09, Section 4].

Definition 2.1. Let X be a normal projective variety over \mathbb{C} with at worst \mathbb{Q} -Gorenstein log-terminal singularities. Denote by r the minimal positive integer such that rK_X is a Cartier divisor. Let $\rho: Y \rightarrow X$ be a log-desingularization together with smooth irreducible divisors D_1, \dots, D_k with simple normal crossings whose support covers the exceptional locus of ρ . We can uniquely write

$$K_Y = \rho^*K_X + \sum_{i=1}^k a_i D_i$$

for some rational numbers $a_i \in \frac{1}{r}\mathbb{Z}$ satisfying the additional condition $a_i = 0$ if D_i is not in the exceptional locus of ρ . We set $I := \{1, \dots, k\}$ and, for any $\emptyset \subseteq J \subseteq I$, we define

$$D_J := \begin{cases} Y & \text{if } J = \emptyset, \\ \bigcap_{j \in J} D_j & \text{if } J \neq \emptyset, \end{cases} \quad D_J^\circ := D_J \setminus \bigcup_{j \in I \setminus J} D_j.$$

Let $\phi: K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow M$ be a $\mathbb{Z}[\lambda]$ -module homomorphism. It induces a $S_{(r)}^{-1}\mathbb{Z}[\lambda]$ -module homomorphism

$$S_{(r)}^{-1}\phi: \widetilde{\mathcal{M}}_{\mathbb{C}}^{(r)} \rightarrow S_{(r)}^{-1}M.$$

We define the *generalized stringy invariant* associated to ϕ as

$$\begin{aligned} \phi^{\text{str}}(X) &:= \sum_{\emptyset \subseteq J \subseteq I} \left(\prod_{j \in J} \frac{\lambda - 1}{\lambda^{a_j+1} - 1} \right) \cdot \phi(D_J^\circ) \\ &= \sum_{\emptyset \subseteq J \subseteq I} \left(\prod_{j \in J} \frac{\lambda - 1}{\lambda^{a_j+1} - 1} - 1 \right) \cdot \phi(D_J) \in S_{(r)}^{-1}M \end{aligned}$$

Theorem 2.2. *The definition of $\phi^{\text{str}}(X)$ is independent of the choice of the log-desingularization.*

Proof. It follows from the theory of motivic integration (see, for instance, [Vey01, Appendix]) that the element

$$[X]^{\text{str}} := \sum_{\emptyset \subseteq J \subseteq I} \left(\prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1} \right) \cdot [D_J^\circ] \in \widetilde{\mathcal{M}}_{\mathbb{C}}^{(r)}$$

is independent of the choice of a log-desingularization. By $\phi^{\text{str}}(X) = S_{(r)}^{-1}\phi([X]^{\text{str}})$, the element $\phi^{\text{str}}(X)$ is also independent of the choice of the log-desingularization. \square

We now consider examples of $\mathbb{Z}[\lambda]$ -module homomorphisms $\phi: K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow M$ and their respective stringy invariants.

Example 2.3. The *virtual Hodge polynomial* (also known as the *E-polynomial*)

$$E(X; u, v) := \sum_{p, q} (-1)^{p+q} h^{p, q}(X) u^p v^q$$

induces a $\mathbb{Z}[\lambda]$ -module homomorphism $K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow \mathbb{Z}[u, v]$ where the $\mathbb{Z}[\lambda]$ -module structure on $\mathbb{Z}[u, v]$ is given by the map $\lambda \mapsto uv$. By Theorem 2.2, we obtain the *stringy E-function*

$$E^{\text{str}}(X; u, v) \in S_{(r)}^{-1}\mathbb{Z}[u, v].$$

Example 2.4. The topological virtual Betti polynomial

$$B_{\text{top}}(X; t) := E(X; t, t)$$

induces a $\mathbb{Z}[\lambda]$ -module homomorphism $K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow \mathbb{Z}[t]$ where the $\mathbb{Z}[\lambda]$ -module structure on $\mathbb{Z}[t]$ is given by the map $\lambda \mapsto t^2$. By Theorem 2.2, we obtain the *topological stringy Betti function*

$$B_{\text{top}}^{\text{str}}(X; t) \in S_{(r)}^{-1}\mathbb{Z}[t^2].$$

Example 2.5. The topological Euler number

$$e_{\text{top}}(X) := B(X; 1) = E(X; 1, 1)$$

induces a $\mathbb{Z}[\lambda]$ -module homomorphism $K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow \mathbb{Z}$ where the $\mathbb{Z}[\lambda]$ -module structure on \mathbb{Z} is given by the map $\lambda \mapsto 1$. By Theorem 2.2, we obtain the *topological stringy Euler number*

$$e_{\text{top}}^{\text{str}}(X) \in S_{(r)}^{-1}\mathbb{Z}.$$

Using the map $S_{(r)}^{-1}\mathbb{Z} \rightarrow \mathbb{Q}$, $\lambda^{1/r} \mapsto 1$, we interpret $e_{\text{top}}^{\text{str}}(X) \in \mathbb{Q}$, and then we have

$$e_{\text{top}}^{\text{str}}(X) = \lim_{t \rightarrow 1} B_{\text{top}}^{\text{str}}(X; t) = \lim_{u, v \rightarrow 1} E^{\text{str}}(X; u, v).$$

The following was shown by Teh using [Bit04, Theorem 3.1].

Proposition 2.6 ([Teh09, Proposition 4, $k = 2p$]). *The algebraic Betti polynomial of smooth projective algebraic varieties*

$$B_{\text{alg}}(X; t) := \sum_i b_{\text{alg}}^{2i}(X) t^{2i}$$

can be extended to a $\mathbb{Z}[\lambda]$ -module homomorphism

$$B_{\text{alg}}: K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow \mathbb{Z}[t^2]$$

where the $\mathbb{Z}[\lambda]$ -module structure on $\mathbb{Z}[t^2]$ is given by $\lambda \mapsto t^2$.

Corollary 2.7. *By Theorem 2.2, we obtain that the algebraic stringy Betti function*

$$B_{\text{alg}}^{\text{str}}(X; t) \in S_{(r)}^{-1}\mathbb{Z}[t]$$

does not depend on the choice of a log-desingularization and the same is true for the algebraic stringy Euler

$$e_{\text{alg}}^{\text{str}}(X) = \lim_{t \rightarrow 1} B_{\text{alg}}^{\text{str}}(X; t) \in \mathbb{Q}.$$

Example 2.8. It is also possible to consider the following modified version of the virtual E -polynomial:

$$E_{\text{even}}(X; u, v) := \frac{1}{2} (E(X; u, v) + E(X; -u, -v)) = \sum_{p+q \in 2\mathbb{Z}} h^{p,q}(X) u^p v^q.$$

The corresponding stringy invariant

$$E_{\text{even}}^{\text{str}}(X; u, v) := \sum_{\emptyset \subseteq J \subseteq I} E_{\text{even}}(D_J^{\circ}; u, v) \left(\prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \right)$$

we call the *even stringy E -function*.

Example 2.9. Another modified version of the virtual E -polynomial is

$$E_{p=q}(X; u, v) := \sum_{0 \leq p \leq \dim X} h^{p,p}(X) u^p v^p.$$

This leads to the stringy invariant

$$E_{p=q}^{\text{str}}(X; u, v) := \sum_{\emptyset \subseteq J \subseteq I} E_{p=q}(D_J^{\circ}; u, v) \left(\prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \right).$$

Remark 2.10. We also obtain the modified Euler numbers

$$e_{\text{even}}^{\text{str}}(X) := \lim_{u,v \rightarrow 1} E_{\text{even}}^{\text{str}}(X; u, v) \quad \text{and} \quad e_{p=q}^{\text{str}}(X) := \lim_{u,v \rightarrow 1} E_{p=q}^{\text{str}}(X; u, v).$$

The results of the next section show that one can expect that Conjectures 1.1, 1.5, and 1.6 could also be true for the rational numbers $e_{\text{even}}^{\text{str}}$ and $e_{p=q}^{\text{str}}$, i.e., these alternative stringy invariants are also candidates for invariants attaining their minimum on the minimal model in a given birational class.

3. EQUIVARIANT DESINGULARIZATIONS WITH FINITELY MANY ORBITS

Let G be a connected linear algebraic group. We consider the category of algebraic G -varieties with G -equivariant birational maps over \mathbb{C} . We need two results of Brion and Peyre.

Lemma 3.1 ([BP02, Lemma 1 and Lemma 2]). *Let $H \subseteq G$ be a closed subgroup. Then there exists a locally trivial fibration in the Zariski topology $G/H \rightarrow Z$ with fiber T such that $B_{\text{alg}}(Z; t) = B_{\text{top}}(Z; t)$ and T is an algebraic torus.*

Proposition 3.2. *Let $H \subseteq G$ be a closed subgroup. Then for the quasi-projective variety G/H one has*

$$B_{\text{alg}}(G/H; t) = B_{\text{top}}(G/H; t).$$

In particular, one has $e_{\text{alg}}(G/H) = e_{\text{top}}(G/H)$.

Proof. Using the fibration from Lemma 3.1, one obtains

$$B_{\text{alg}}(G/H; t) = B_{\text{alg}}(T; t) B_{\text{alg}}(Z; t) = B_{\text{top}}(T; t) B_{\text{top}}(Z; t) = B_{\text{top}}(G/H; t)$$

because $B_{\text{alg}}(T; t) = B_{\text{top}}(T; t) = (t^2 - 1)^{\dim T}$. \square

Proposition 3.3 ([BP02, Theorem 1(b)]). *Let $H \subseteq G$ be a closed subgroup. Then one has*

$$e_{\text{top}}(G/H) = \begin{cases} 0 & \text{if } \text{rank } G > \text{rank } H, \\ |W_G|/|W_H| & \text{otherwise,} \end{cases}$$

where W_G (resp. W_H) denotes the Weyl group of G^{red} (resp. of $(H^\circ)^{\text{red}}$). In particular, one has $e_{\text{top}}(G/H) > 0$ if H is a parabolic subgroup.

Theorem 3.4. *Let X be a \mathbb{Q} -Gorenstein projective G -variety with at worst log-terminal singularities. Assume that there exists a G -equivariant log-desingularization $\rho: Y \rightarrow X$ such that Y contains only finitely many G -orbits. Then $e_{\text{alg}}^{\text{str}}(X) > 0$.*

Proof. We write

$$K_Y = \rho^* K_X + \sum_{i=1}^k a_i D_i$$

for some G -invariant smooth irreducible divisors D_1, \dots, D_k . Using Definition 2.1, we obtain

$$e_{\text{alg}}^{\text{str}}(X) = \sum_{\emptyset \subseteq J \subseteq I} e_{\text{alg}}(D_J^\circ) \prod_{j \in J} \frac{1}{a_j + 1}.$$

Since the G -invariant locally closed subvariety D_J° is a union of finitely many G -orbits, by Proposition 3.3, we obtain $e_{\text{alg}}(D_J^\circ) \geq 0$ for all $J \subseteq I$. Together with the inequalities $a_j + 1 > 0$ for all $j \in I$ it implies $e_{\text{alg}}^{\text{str}}(X) \geq 0$. Choose a subset $J \subseteq I$ with $D_J^\circ = D_J$. As D_J° contains a closed G -orbit, by Proposition 3.3, we obtain $e_{\text{alg}}(D_J^\circ) = e_{\text{alg}}(D_J) > 0$ and, hence, the strict inequality $e_{\text{alg}}^{\text{str}}(X) > 0$. \square

Theorem 3.5. *Let $g: X \dashrightarrow X^+$ be a G -equivariant Mori flip with G -equivariant log-desingularizations $\rho: Y \rightarrow X$ and $\rho_+: Y \rightarrow X^+$ as well as a G -equivariant commutative diagram*

$$\begin{array}{ccc} & Y & \\ \rho \swarrow & & \searrow \rho_+ \\ X & \dashrightarrow^g & X^+ \end{array}$$

such that Y contains only finitely many G -orbits. Then we have $e_{\text{alg}}^{\text{str}}(X) > e_{\text{alg}}^{\text{str}}(X^+)$.

Proof. Write

$$K_Y = \rho^* K_X + \sum_{i=1}^k a_i D_i = \rho_+^* K_{X^+} + \sum_{i=1}^k a_i^+ D_i$$

for some G -invariant prime divisors D_1, \dots, D_k . By Definition 2.1, we have

$$\begin{aligned} e_{\text{alg}}^{\text{str}}(X) &= \sum_{\emptyset \subseteq J \subseteq I} e_{\text{alg}}(D_J^\circ) \prod_{j \in J} \frac{1}{a_j + 1}, \\ e_{\text{alg}}^{\text{str}}(X^+) &= \sum_{\emptyset \subseteq J \subseteq I} e_{\text{alg}}(D_J^\circ) \prod_{j \in J} \frac{1}{a_j^+ + 1}. \end{aligned}$$

According to [IS05, Lemma 3.4], we have $a_j \leq a_j^+$ for every $j \in I$ and there exists $j_0 \in I$ such that $a_{j_0} < a_{j_0}^+$. Since the G -invariant locally closed subvariety D_J° is a union of finitely many G -orbits, by Proposition 3.3, we have $e_{\text{alg}}(D_J^\circ) \geq 0$ for all subsets $J \subseteq I$. Thus, we obtain $e_{\text{alg}}^{\text{str}}(X) \geq e_{\text{alg}}^{\text{str}}(X^+)$. Choose a subset $J \subseteq I$ with $j_0 \in J$ and $D_J^\circ = D_J$. As D_J° contains a projective G -orbit, by Proposition 3.3, one obtains $e_{\text{alg}}(D_J^\circ) > 0$ and, hence, the strict inequality $e_{\text{alg}}^{\text{str}}(X) > e_{\text{alg}}^{\text{str}}(X^+)$. \square

Theorem 3.6. *Let $f: X \rightarrow X'$ be a G -equivariant divisorial Mori contraction with a G -equivariant log-desingularization $\rho: Y \rightarrow X$ such that Y contains only finitely many G -orbits. Then $e_{\text{str}}^{\text{alg}}(X) > e_{\text{str}}^{\text{alg}}(X')$.*

Proof. We write

$$K_Y = \rho^*(K_X) + \sum_{i=1}^k a_i D_i = \rho^* f^* K_{X'} + \sum_{i=1}^k a_i' D_i$$

for some G -invariant prime divisors D_1, \dots, D_k . By Definition 2.1, we have

$$\begin{aligned} e_{\text{alg}}^{\text{str}}(X) &= \sum_{\emptyset \subseteq J \subseteq I} e_{\text{alg}}(D_J^\circ) \prod_{j \in J} \frac{1}{a_j + 1}, \\ e_{\text{alg}}^{\text{str}}(X') &= \sum_{\emptyset \subseteq J \subseteq I} e_{\text{alg}}(D_J^\circ) \prod_{j \in J} \frac{1}{a_j' + 1}. \end{aligned}$$

Again we have $a_j \leq a_j'$ for every $j \in I$ and there exists $j_0 \in I$ such that $a_{j_0} < a_{j_0}'$. The rest of the proof is the same as in the proof of Theorem 3.5. \square

Remark 3.7 (Proof of Theorem 1.8). Spherical varieties have finitely many G -orbits, and the desingularizations of spherical varieties required in Theorems 3.4, 3.5, and 3.6 can be always constructed explicitly using subdivisions of the corresponding colored fans.

Remark 3.8. As in [Bat99b], it is possible to generalize Definition 2.1 to define stringy invariants for any Kawamata log-terminal pair (X, Δ) .

For example, let $\rho: Y \rightarrow X$ be a log-desingularization of (X, Δ) together with smooth irreducible divisors D_1, \dots, D_k with simple normal crossings whose support covers the exceptional locus of ρ and the support of $\rho^*\Delta$. We can uniquely write

$$K_Y = \rho^*(K_X + \Delta) + \sum_{i=1}^k a_i D_i$$

for some rational numbers $a_i \in \frac{1}{r}\mathbb{Z}$ satisfying the additional condition that $-a_i$ is the multiplicity of D_i in Δ if D_i is not in the exceptional locus of ρ . We then define

$$e_{\text{alg}}^{\text{str}}(X, \Delta) := \sum_{\emptyset \subseteq J \subseteq I} e_{\text{alg}}^{\text{str}}(D_J^\circ) \prod_{j \in J} \frac{1}{a_j + 1} \in \mathbb{Q}.$$

Conjecture 3.9. *The algebraic stringy Euler number $e_{\text{alg}}^{\text{str}}(X, \Delta)$ is strictly monotone with respect to the elementary birational transformations from the Mori theory of Kawamata log-terminal pairs (X, Δ) .*

It is possible to prove generalized versions of Theorems 3.4, 3.5, and 3.6 for Kawamata log-terminal pairs (X, Δ) where the irreducible components of Δ are G -invariant. In particular, we obtain the following result.

Theorem 3.10. *The algebraic stringy Euler number $e_{\text{alg}}^{\text{str}}(X, \Delta)$ is strictly monotone with respect to the elementary G -equivariant birational transformations from the G -equivariant Mori theory of Kawamata log-terminal pairs (X, Δ) for projective spherical varieties X and G -invariant divisors Δ .*

Remark 3.11. Let $B \subseteq G$ be a Borel subgroup. Since every divisor on a spherical variety is linearly equivalent to a B -invariant divisor, it is natural to consider the B -equivariant Mori theory of pairs (X, Δ) on spherical G -varieties X . Irreducible B -invariant divisors on X that are not G -invariant are called *colors*. As in the case of toric varieties, B -invariant Cartier divisors can be described by piecewise linear functions on the colored fan of a spherical variety. It is possible that the irreducible B -invariant divisors (colors) have singularities, even in the open G -orbit, and even if all colors are smooth, they may not satisfy the normal crossing condition. The B -equivariant log-desingularizations of B -invariant divisors on horospherical varieties have been constructed by Pasquier in [Pas15b] using Bott-Samelson resolutions of Schubert cells in homogeneous spaces. We remark that every spherical variety X contains only finitely many B -orbits. However, a B -equivariant log-desingularization of B -invariant pairs on a spherical variety X having only finitely many B -orbits may not exist in general.

Proposition 3.12. *Consider the spherical $\text{SL}(2)$ -variety $X := \mathbb{P}(\mathfrak{sl}_2) \cong \mathbb{P}^2$, i.e., the projectivization of the adjoint representation of $\text{SL}(2)$. Then the set of B -invariant*

divisors in X does not admit a B -equivariant log-desingularization with finitely many B -orbits.

Proof. It is easy to show that X contains exactly two irreducible B -invariant divisors D_0 and D_1 . The divisor $D_0 \cong \mathbb{P}^1$ is an $\mathrm{SL}(2)$ -invariant conic in \mathbb{P}^2 . It is the projectivization of the set of nilpotent matrices in \mathfrak{sl}_2 . The B -invariant divisor $D_1 \cong \mathbb{P}^1$ is the tangent line to the conic D_0 at the B -fixed point $x_B \in D_0$.

Assume that there exists a B -equivariant desingularization $\rho: Y \rightarrow X$ such that Y has only finitely many B -orbits and the set of B -invariant divisors in Y has simple normal crossings. Then the birational morphism $\rho: Y \rightarrow X$ between two smooth projective algebraic surfaces Y and X can be decomposed into a sequence of B -equivariant blow-downs $Y =: X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 := X$. Moreover, every smooth projective algebraic surface X_i must have finitely many B -orbits. Since X_0 has only one B -fixed point $x_0 := D_0 \cap D_1$, the birational morphism $\rho_1: X_1 \rightarrow X_0$ must be the blow-up of this point. Then the B -variety X_1 contains three B -invariant divisors $\rho^{-1}(D_0)$, $\rho^{-1}(D_1)$, and $\rho^{-1}(x_0)$ having a common B -fixed point $x_1 \in X_1$. It is easy to show that x_1 is the unique B -fixed point in X_1 . Therefore, the birational B -equivariant morphism $\rho_2: X_2 \rightarrow X_1$ must be the blow-up of x_1 . On the other hand, the tangent spaces of three divisors at the point x_1 are pairwise different. Hence, the group B has at least 3 pairwise distinct fixed points in the exceptional divisor $E := \rho^{-1}(x_1) \subseteq X_2$. So B must act trivially on E , i.e., there exist infinitely many B -orbits in X_2 , a contradiction. \square

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